Chapter 4 More Derivatives

Chain Rule

Derivative of a Composite Function

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (f \circ g)' = f'(g(x)) \cdot g'(x)$$

EXAMPLE 2 Relating Derivatives

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$y = (3x^2 + 1)^2$$

$$y = u^2 \qquad u = 3x^2 + 1$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$2u \cdot 6x$$

$$2(3x^2+1)6x$$

$$12x(3x^2+1)$$

$$\frac{dy}{dx} \quad 36x^3 + 12x$$

RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

ex. $y = \sin(3x^2 + 1)$ $u = 3x^2 + 1$

$y = \sin u$ $u = 3x^2 + 1$

$\frac{dy}{du} = \cos u$ $\frac{du}{dx} = 6x$

$\cos u \cdot 6x$

$\cos(3x^2+1)6x$

$6x\cos(3x^2+1)$

ex. $y = \sin(3x^2 + 1)$ $u = 3x^2 + 1$

$$\frac{dy}{dx} = \cos(3x^2+1)6x$$

$$= 6x\cos(3x^2+1)$$

ex. $y = (x^3 + 3x^2)^4$ $u = x^3 + 3x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$y = u^4$ $u = x^3 + 3x^2$

$$\frac{dy}{du} = 4u^3 \cdot (3x^2 + 6x)$$

$$= 4(x^3+3x^2)^3(3x^2+6x)$$

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$$f(x) = (4x^3+5x)^7$$

$$f'(x) = 7(4x^3+5x)^6(12x^2+5)$$

Example: $f(g(x))=\sin(x^2-4)$

EXAMPLE 5 A Three-Link "Chain"

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$$g(t) = \tan(5 - \sin 2t)$$

$$g'(t) = \sec^2(5 - \sin 2t)(-\cos 2t)2$$

$$-2\cos 2t \sec^2(5 - \sin 2t)$$

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

EXAMPLE 7 Finding Slope

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Multiple Choice Which of the following is dy/dx if $y = \cos^2 (x^3 + x^2)$?

(A) $-2(3x^2 + 2x)$

(B) $-(3x^2 + 2x) \cos (x^3 + x^2) \sin (x^3 + x^2)$

(C) $-2(3x^2 + 2x) \cos (x^3 + x^2) \sin (x^3 + x^2)$

(D) $2(3x^2 + 2x) \cos (x^3 + x^2) \sin (x^3 + x^2)$

(E) $2(3x^2 + 2x)$

Multiple Choice Which of the following is dy/dx if $y = \tan(4x)$? E

(A) $4\sec(4x)\tan(4x)$ (B) $\sec(4x)\tan(4x)$ (C) $4\cot(4x)$

(D) $\sec^2(4x)$ (E) $4\sec^2(4x)$

2. Which of the following gives $\frac{dy}{dx}$ if $y = \cos^3(3x-2)$?

(A) $-9\cos^2(3x-2)\sin(3x-2)$

(B) $-3\cos^2(3x-2)\sin(3x-2)$

(C) $9\cos^2(3x-2)\sin(3x-2)$

(D) $-9\cos^2(3x-2)$

(E) $-3\cos^2(3x-2)$

56. ***Working with Numerical Values*** Suppose that functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f(x)$ at $x = 2$ **(b)** $f(x) + g(x)$ at $x = 3$

(c) $f(x) \cdot g(x)$ at $x = 3$ **(d)** $f(x)/g(x)$ at $x = 2$

(e) $f(g(x))$ at $x = 2$ **(f)** $\sqrt{f(x)}$ at $x = 2$

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